Nested Radicals

Radical:

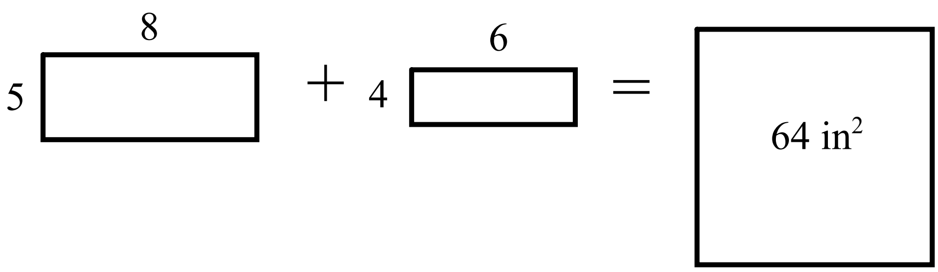
In **mathematics**, a **radical** expression is **defined** as any expression containing a **radical** (√) symbol. Many people mistakenly call this a 'square root' symbol, and many times it is used to determine the square root of a number. However, it can also be used to describe a cube root, a fourth root or higher.

Example of Middle School Usage

Sheet Metal Trades A square hot-air duct must have the same area as two rectangular ones 5 in. × 8 in. and 4 in. × 6 in. How long are the sides of the square duct?

$$(5 \times 8) + (4 \times 6) = 40 \text{ in.}^2 + 24 \text{ in.}^2 = 64 \text{ in.}^2$$

$$\sqrt{64 \text{ in.}^2} = 8 \text{ in.}$$



Usage in Beginning of High School Math

Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

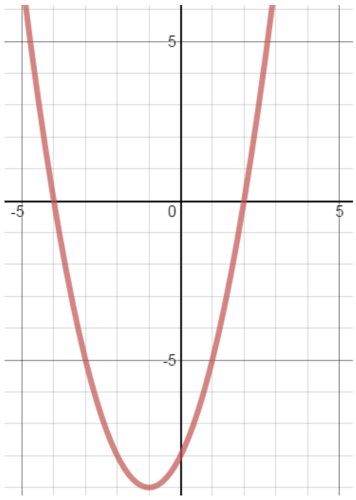
$$\boxed{x^2 + 2x - 8 = 0}$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-8)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{4 + 32}}{2} = \frac{-2 \pm \sqrt{36}}{2}$$

$$= \frac{-2 \pm 6}{2}$$

$$x = -4; \quad x = 2$$



François Viète

1540 – 23 February 1603

Usage in Advanced Mathematics

At the time Viète published his formula, methods for approximating π to (in principle) arbitrary accuracy had long been known. Viète's own method can be interpreted as a variation of an idea of Archimedes of approximating the area of a circle by that of a many-sided polygon, used by Archimedes to find the approximation.

However, by publishing his method as a mathematical formula, Viète formulated the first instance of an infinite product known in mathematics, and the first example of an explicit formula for the exact value of π. As the first formula representing a number as the result of an infinite process rather than of a finite calculation, Viète's formula has been noted as the beginning of mathematical analysis and even more broadly as "the dawn of modern mathematics".

Viete's Formula for Pi

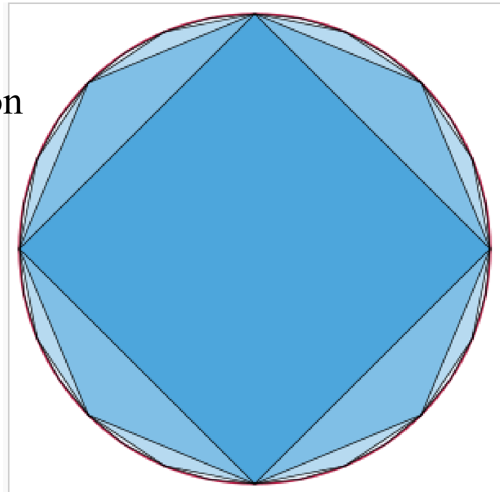
$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdots$$

ratio of square and octagon

ratio of octagon and 16-gon

Viète obtained his formula by comparing the areas of regular polygons with 2^n and 2^{n+1} sides inscribed in a circle.[1][6] The first term in the product, $\frac{\sqrt{2}}{2}$, is the ratio of areas of a square and an octagon, the second term is the ratio of areas of an octagon and a hexadecagon, etc. Thus, the product telescopes to give the ratio of areas of a square (the initial polygon in the sequence) to a circle (the limiting case of a 2^n

$$\frac{\text{Area of a Square}}{\text{Area of a Circle}} = \frac{2r^2}{\pi r^2}$$



A sequence of regular polygons with numbers of sides equal to powers of two, inscribed in a circle. The ratios between areas or perimeters of consecutive polygons in the sequence give the terms of Viète's formula.

Viete's Formula for Pi

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdots .$$

Lead to usage in trigonometric values

$$\sin\left(\frac{\pi}{8}\right) = \tfrac{1}{2}\sqrt{2-\sqrt{2}}$$

$$\cos\left(\frac{\pi}{8}\right) = \tfrac{1}{2}\sqrt{2+\sqrt{2}}$$

$$\sin\left(\frac{\pi}{16}\right) = \tfrac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}}$$

$$\cos\left(\frac{\pi}{16}\right) = \tfrac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}} \; .$$

The Curiosity for Nested Radicals is Born

$$\sqrt{3\sqrt{3\sqrt{3\sqrt{3\sqrt{3\ldots}}}}} = ?\ x$$

$$\sqrt{3x} = x$$

$$3x = x^2$$

$$3 = x$$

The Curiosity for Nested Radicals is Born

$$\sqrt{3\sqrt{3\sqrt{3\sqrt{3\sqrt{3\ldots}}}}} = ?$$

A look with a different lens

$$\sqrt{3\sqrt{3\sqrt{3\sqrt{3\sqrt{3\ldots}}}}} = 3^{1/2}3^{1/4}3^{1/8}\ldots$$

$$3^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}}$$

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$$

Nested Radicals

With expressions like this, people also became interested in infinitely nested sums.

$$\cos\left(\frac{\pi}{16}\right) = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}\ .$$

$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}}} = ?\ x$$

$$\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}}}_{x}$$

$$\sqrt{2+x} = x$$

$$2+x = x^2$$

$$0 = x^2 - x - 2$$

$$0 = (x-2)(x+1)$$

$$\boxed{x=2}$$

A Surprising Result

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\dots}}}}} = x$$

$$\sqrt{1+x} = x$$

$$1+x = x^2$$

$$0 = x^2 - x - 1$$

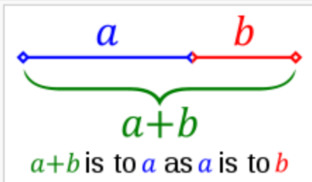
$$x^2 - x - 1 = 0$$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$x = \frac{1 \pm \sqrt{5}}{2}$$

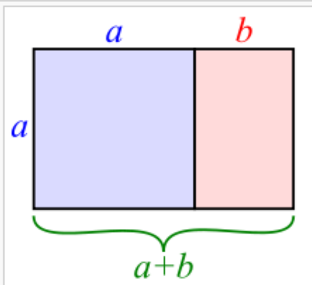
$$x = \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$$

GOLDEN RATIO

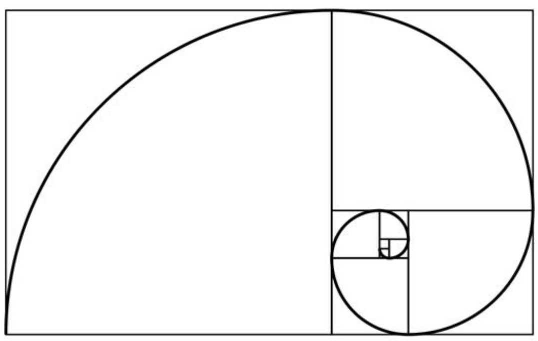


Line segments in the golden ratio

$$\varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots.$$



A golden rectangle (in pink) with longer side **a** and shorter side **b**, when placed adjacent to a square with sides of length **a**, will produce a similar golden rectangle with longer side **a + b** and shorter side **a**. This illustrates the relationship $\frac{a+b}{a} = \frac{a}{b} \equiv \varphi$.



Srinivasa Ramanujan

22 December 1887 – 26 April 1920

During his short life, Ramanujan independently compiled nearly 3,900 results (mostly identities and equations). Nearly all his claims have now been proven correct, although some were already known.

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 396^{4k}}.$$

Srinivasa Ramanujan

22 December 1887 – 26 April 1920

1729

The number 1729 is known as the Hardy–Ramanujan number after a famous anecdote of the British mathematician G. H. Hardy regarding a visit to the hospital to see Ramanujan. In Hardy's words:

I remember once going to see him when he was ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavorable omen. 'No', he replied, 'it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways.'

The two different ways are

$$1729 = 1^3 + 12^3 = 9^3 + 10^3.$$

Suppose that we rate mathematicians on the basis of pure talent on a scale from 0 to 100, Hardy gave himself a score of 25, J.E. Littlewood 30, David Hilbert 80 and Ramanujan 100.

Ramanujan's Nested Radicals (1911)

Ramanujan posed the following problem to the *Journal of Indian Mathematical Society*:

$$? = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + \cdots}}}.$$

After six months, nobody had been able to solve the problem, so Ramanujan revealed the surprisingly simple answer, which is 3.

How Ramanujan did it:

It can then be shown that

$$F(x) = x + n + a.$$

which can be simplified to

$$F(x)^2 = ax + (n+a)^2 + x\sqrt{a(x+n) + (n+a)^2 + (x+n)\sqrt{\cdots}},$$

which can be simplified to

$$F(x)^2 = ax + (n+a)^2 + xF(x+n).$$

So, setting a =0, n = 1, and x = 2, we have

$$3 = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + \cdots}}}.$$

Ramanujan's Nested Radicals

Ramanujan posed the following problem to the *Journal of Indian Mathematical Society*:

$$? = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + \cdots}}}.$$

This was posed to my Intro to Proofs class at the University of Montana.

So, of course, what did I do to start?

Once I knew it was 3. I worked backwards.

$$3 = \sqrt{9}$$

$$= \sqrt{1 + 8}$$

$$= \sqrt{1 + 2(4)}$$

$$= \sqrt{1 + 2\sqrt{1 + 3(5)}}$$

THE END


√A
whenever i get to math class, i have a radical expression on my face.